

Dirac Operators with Torsion and the Noncommutative Residue for Manifolds with Boundary

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Abstract

In this paper, we get the Kastler-Kalau-Walze theorem associated to Dirac operators with torsion on compact manifolds with boundary. We give two kinds of operator-theoretic explanations of the gravitational action in the case of 4-dimensional compact manifolds with flat boundary. Furthermore, we get the Kastler-Kalau-Walze type theorem for four dimensional complex manifolds associated with nonminimal operators.

Keywords: Dirac Operators with torsion; noncommutative residue; orthogonal connection with torsion; gravitational action for manifolds with boundary.

1. Introduction

The Dirac operators evolved into an important tool of modern mathematics, occurring for example in index theory, gauge theory, geometric quantization, etc. Recently, Dirac operators have assumed a significant place in Connes' noncommutative geometry in [1] as the main ingredient in the definition of a K-cycle. Thus disguised, Dirac operators re-enter modern physics, since non-commutative geometry can be used, e.g. to derive the action of the Standard Model of elementary particles, as shown in [2, 3]. The noncommutative residue found in [4] and [5] plays a prominent role in noncommutative geometry.

In [1], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Further, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportion to the Einstein-Hilbert action in [6], which we call the Kastler-Kalau-Walze theorem. In [7], Kastler gave a brute-force proof of this theorem. In [8], Kalau and Walze proved this theorem in the normal coordinates system simultaneously. And then, Ackermann gave a note on a new proof of this theorem by means of the heat kernel expansion in [9]. For 3, 4-dimensional spin manifolds with boundary, Wang proved a Kastler-Kalau-Walze type theorem for the Dirac operator and the signature operator in [10]. In [11], Wang computed the lower dimensional volume $\text{Vol}^{(2,2)}$ for 5-dimensional and 6-dimensional spin manifolds with boundary and also got the Kastler-Kalau-Walze type theorem in this case. We proved the Kastler-Kalau-Walze type theorems for foliations with or without boundary associated with sub-Dirac operators for foliations in [12]. In [13], Ackermann and Tolksdorf proved a generalized version of the well-known Lichnerowicz formula for the square of the most general Dirac operator with torsion D_T on an even-dimensional spin manifold associated to a metric connection with torsion. Recently, Pfäffle and Stephan considered compact Riemannian spin manifolds without boundary equipped with orthogonal connections, and investigated the induced Dirac operators in [14]. In [15], Pfäffle and Stephan considered orthogonal connections with arbitrary torsion on compact Riemannian manifolds, and for the induced Dirac operators, twisted Dirac operators and Dirac operators of Chamseddine-Connes type they computed the spectral action.

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The purpose of this paper is to generalize the results in [10], [14] and get a Kastler-Kalau-Walze type theorems associated with Dirac operators with torsion on compact manifolds with boundary. We derive the gravitational action on boundary by the noncommutative residue associated with Dirac operators with torsion. For lower dimensional compact Riemannian manifolds with boundary and complex manifolds with boundary, we compute the lower dimensional volume $\widetilde{\text{Wres}}[\pi^+(D_T^*)^{-p_1} \circ \pi^+ D_T^{-p_2}]$, and we get the Kastler-Kalau-Walze theorem for lower dimensional manifolds with boundary. On the other hand, a special case of Dirac operators with torsion is the Dolbeault operator for complex manifolds. In [16], we considered the nonminimal operators as generalization of De-Rham Hodge operators and got the Kastler-Kalau-Walze type theorems for nonminimal operators. In this paper, we consider the complex analogy of nonminimal operators and prove a Kastler-Kalau-Walze type theorems for complex nonminimal operators.

This paper is organized as follows: In Section 2, we define lower dimensional volumes of spin manifolds with torsion. In Section 3, for 4-dimensional spin manifolds with boundary and the associated Dirac operators with torsion D_T^*, D_T , we compute the lower dimensional volume $\text{Vol}_4^{(1,1)}$ and get a Kastler-Kalau-Walze type theorem in this case. In Section 4, two kinds of operator theoretic explanations of the gravitational action for boundary in the case of 4-dimensional manifolds with boundary will be given. In Section 5 and Section 6, we get the Kastler-Kalau-Walze type theorem for 6-dimensional spin manifolds with boundary associated to $(D_T^*)^2, D_T^2$ with torsion and 4-dimensional spin manifolds with boundary for the operator $P^+ D_T^* D_T$. In Section 7, we investigate 4-dimensional complex manifolds without boundary associated with complex nonminimal operators.

2. Lower-Dimensional Volumes of Spin Manifolds with Torsion

In this section we consider an n -dimensional oriented Riemannian manifold (M, g^M) equipped with some spin structure. The Levi-Civita connection $\nabla : \Gamma(TM) \rightarrow \Gamma(T^*M \otimes TM)$ on M induces a connection $\nabla^S : \Gamma(S) \rightarrow \Gamma(T^*M \otimes S)$. By adding a additional torsion term $t \in \Omega^1(M, \text{End}TM)$ we obtain a new covariant derivative

$$\tilde{\nabla} := \nabla + t \quad (2.1)$$

on the tangent bundle TM . Since t is really a one-form on M with values in the bundle of skew endomorphism $Sk(TM)$ in [17], ∇ is in fact compatible with the Riemannian metric g and therefore also induces a connection $\tilde{\nabla}^S := \nabla^S + T$ on the spinor bundle. Here $T \in \Omega^1(M, \text{End}S)$ denotes the ‘lifted’ torsion term $t \in \Omega^1(M, \text{End}TM)$.

Next, we will briefly discuss the construction of this connection. Again, we write $\tilde{\nabla}_X Y = \nabla_X Y + A(X, Y)$ with the Levi-Civita connection ∇ . For any $X \in T_p M$ the endomorphism $A(X, \cdot)$ is skew-adjoint and hence it is an element of $\mathfrak{so}(T_p M)$, we can express it as

$$A(X, \cdot) = \sum_{i < j} \alpha_{ij} E_i \wedge E_j. \quad (2.2)$$

Here $E_i \wedge E_j$ is meant as the endomorphism of $T_p M$ defined by $E_i \wedge E_j$. For any $X \in T_p M$ one determines the coefficients in (2.2) by

$$\alpha_{ij} = \langle A(X, E_i), E_j \rangle = A_{X E_i E_j}. \quad (2.3)$$

Each $E_i \wedge E_j$ lifts to $\frac{1}{2} E_i \cdot E_j$ in $\text{spin}(n)$, and the spinor connection induced by $\tilde{\nabla}$ is locally given by

$$\tilde{\nabla}_X \psi = \nabla_X \psi + \frac{1}{2} \sum_{i < j} \alpha_{ij} E_i \cdot E_j \psi = \nabla_X \psi + \frac{1}{2} \sum_{i < j} A_{X E_i E_j} E_i \cdot E_j \psi. \quad (2.4)$$

The connection given by (2.4) is compatible with the metric on spinors and with Clifford multiplication.

Then, the Dirac operator associated to the spinor connection from (2.4) is defined as

$$\begin{aligned} D_T \psi &= \sum_{i=1}^n E_i \tilde{\nabla}_{E_i} \psi = D\psi + \frac{1}{2} \sum_{i=1}^n \sum_{j < k} A_{E_i E_j E_k} E_i \cdot E_j \cdot E_k \psi \\ &= D\psi + \frac{1}{4} \sum_{i,j,k=1}^n A_{E_i E_j E_k} E_i \cdot E_j \cdot E_k \psi \end{aligned} \quad (2.5)$$

where D is the Dirac operator induced by the Levi-Civita connection and “ \cdot ” is the Clifford multiplication. As Clifford multiplication by any 3-form is self-adjoint we have

$$D_T \psi = D\psi + \frac{3}{2} T \cdot \psi - \frac{n-1}{2} V \cdot \psi, \quad D_T^* \psi = D\psi + \frac{3}{2} T \cdot \psi + \frac{n-1}{2} V \cdot \psi. \quad (2.6)$$

Using the fact that the Clifford multiplication by the vector field V is skew-adjoint, the hermitian product on the spinor bundle one observes that D_T is symmetric with respect to the natural L^2 -scalar product on spinors if and only if the vectorial component of the torsion vanishes, $V \equiv 0$. Note that the Cartan type torsion S does not contribute to the Dirac operator D_T . As $D_T^* D_T$ is a generalized Laplacian, one has the following Lichnerowicz formula.

Theorem 2.1. [14] *For the Dirac operator D_T associated to the orthogonal connection $\tilde{\nabla}$, we have*

$$\begin{aligned} D_T^* D_T \psi &= \Delta \psi + \frac{1}{4} R^g \psi + \frac{3}{2} dT \cdot \psi - \frac{3}{4} \|T\|^2 \psi \\ &\quad + \frac{n-1}{2} \operatorname{div}^g(V) \psi + \left(\frac{n-1}{2}\right)^2 (2-n) |V|^2 \psi \\ &\quad + 3(n-1)(T \cdot V \cdot \psi + (V \lrcorner T) \cdot \psi), \end{aligned} \quad (2.7)$$

for any spinor field ψ , where Δ is the Laplacian associated to the connection

$$\tilde{\nabla}_X \psi = \nabla_X \psi + \frac{3}{2} (X \lrcorner T) \cdot \psi - \frac{n-1}{2} V \cdot X \cdot \psi - \frac{n-1}{2} \langle V, X \rangle \psi. \quad (2.8)$$

To define lower dimensional volume $\operatorname{Vol}_n^{p_1, p_2} M := \widetilde{\operatorname{Wres}}[\pi^+(D_T^*)^{-p_1} \circ \pi^+ D_T^{-p_2}]$, some basic facts and formulae about Boutet de Monvel’s calculus can be find in Sec.2 in [18]. Let M be an n -dimensional compact oriented manifold with boundary ∂M . We assume that the metric g^M on M has the following form near the boundary

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \quad (2.9)$$

where $g^{\partial M}$ is the metric on ∂M . Let $U \subset M$ be a collar neighborhood of ∂M which is diffeomorphic $\partial M \times [0, 1)$. By the definition of $C^\infty([0, 1))$ and $h > 0$, there exists $\tilde{h} \in C^\infty((-\varepsilon, 1))$ such that $\tilde{h}|_{[0, 1)} = h$ and $\tilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric \hat{g} on $\hat{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$\hat{g} = \frac{1}{\tilde{h}(x_n)} g^{\partial M} + dx_n^2, \quad (2.10)$$

such that $\hat{g}|_M = g$. We fix a metric \hat{g} on the \hat{M} such that $\hat{g}|_M = g$. Note D_T is the most general Dirac operator on the spinor bundle S corresponding to a metric connection $\tilde{\nabla}$ on TM . Let p_1, p_2 be nonnegative integers and $p_1 + p_2 \leq n$. From Sec 2.1 of [10], we have

Definition 2.2. *Lower-dimensional volumes of spin manifolds with boundary with torsion are defined by*

$$\operatorname{Vol}_n^{\{p_1, p_2\}} M := \widetilde{\operatorname{Wres}}[\pi^+(D_T^*)^{-p_1} \circ \pi^+ D_T^{-p_2}]. \quad (2.11)$$

Denote by $\sigma_l(A)$ the l -order symbol of an operator A . An application of (2.1.4) in [18] shows that

$$\widetilde{Wres}[\pi^+(D_T^*)^{p_1} \circ \pi^+ D_T^{p_2}] = \int_M \int_{|\xi|=1} \text{trace}_{S(TM)} [\sigma_{-n}((D_T^*)^{-p_1} \circ D_T^{-p_2})] \sigma(\xi) dx + \int_{\partial M} \Phi, \quad (2.12)$$

where

$$\begin{aligned} \Phi &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+\ell}}{\alpha!(j+k+1)!} \text{trace}_{S(TM)} [\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+((D_T^*)^{-p_1})(x', 0, \xi', \xi_n) \\ &\quad \times \partial_{x_n}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(D_T^{-p_2})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \quad (2.13)$$

and the sum is taken over $r - k + |\alpha| + \ell - j - 1 = -n, r \leq -p_1, \ell \leq -p_2$.

3. The Kastler-Kalau-Walze theorem for 4-dimensional spin manifolds with boundary about Dirac Operators with torsion D_T^*, D_T

In this section, we compute the lower dimensional volume for 4-dimension compact manifolds with boundary and get a Kastler-Kalau-Walze type formula in this case.

From now on we always assume that M carries a spin structure so that the spinor bundle is defined and so are Dirac operator, twisted or generalised Dirac operators on M . Connes spectral action principle in [6] states that one can extract any action functional of interest in physics from the spectral data of a Dirac operator.

In the following we consider various Dirac operators D_T induced by orthogonal connections with general torsion as in (2.4). We will consider $D_T^* D_T$ (since D_T is not selfadjoint in general) and the corresponding Seeley-deWitt coefficients. The Chamseddine-Connes spectral action of $D_T^* D_T$ is determined if one knows the second and the fourth Seeley-deWitt coefficient. Since $[\sigma_{-n}((D_T^*)^{-p_1} \circ D_T^{-p_2})]|_M$ has the same expression as $[\sigma_{-n}((D_T^*)^{-p_1} \circ D_T^{-p_2})]|_M$ in the case of manifolds without boundary, so locally we can use the computations Proposition 3.1 in [15] to compute the first term.

Theorem 3.1. [15] *Let M be a 4-dimensional compact manifold without boundary and $\tilde{\nabla}$ be an orthogonal connection with torsion. Then we get the volumes associated to $D_T^* D_T$ on compact manifolds without boundary*

$$Wres((D_T^* D_T)^{-1}) = -\frac{1}{48\pi^2} \int_M \tilde{R}(x) dx, \quad (3.1)$$

where $\tilde{R} = R + 18 \text{div}(V) - 54|V|^2 - 9 \|T\|^2$ and $\int_M \text{div}(V) dV \text{ol}_M = - \int_{\partial M} g(n, V) dV \text{ol}_{\partial M}$.

Theorem 3.2. [15] *Let M be a 6-dimensional compact manifold and $\tilde{\nabla}$ be an orthogonal connection with torsion. Then we get the volume associated to $D_T^* D_T$ on compact manifolds without boundary*

$$Wres((D_T^* D_T)^{-1}) = \frac{11}{720} \mathcal{X}(M) - \frac{1}{360\pi^2} \int_M \|C^g\|^2 dx - \frac{3}{32\pi^2} \int_M (\|\delta T\|^2 + \|d(V)\|^2) dx, \quad (3.2)$$

where C^g denote the Weyl curvature of the Levi-Civita connection and $\mathcal{X}(M)$ denote the Euler characteristic of M .

So we only need to compute $\int_{\partial M} \Phi$. Let $n = 4$, our computation extends to general n . Let

$$F : L^2(\mathbf{R}_t) \rightarrow L^2(\mathbf{R}_v); F(u)(v) = \int e^{-ivt} u(t) dt$$

denote the Fourier transformation and $\Phi(\overline{\mathbf{R}^+}) = r^+ \Phi(\mathbf{R})$ (similarly define $\Phi(\overline{\mathbf{R}^-})$), where $\Phi(\mathbf{R})$ denotes the Schwartz space and

$$r^+ : C^\infty(\mathbf{R}) \rightarrow C^\infty(\overline{\mathbf{R}^+}); f \rightarrow f|_{\overline{\mathbf{R}^+}}; \overline{\mathbf{R}^+} = \{x \geq 0; x \in \mathbf{R}\}. \quad (3.3)$$

We define $H^+ = F(\Phi(\overline{\mathbf{R}^+}))$; $H_0^- = F(\Phi(\overline{\mathbf{R}^-}))$ which are orthogonal to each other. We have the following property: $h \in H^+ (H_0^-)$ iff $h \in C^\infty(\mathbf{R})$ which has an analytic extension to the lower (upper) complex half-plane $\{\text{Im}\xi < 0\}$ ($\{\text{Im}\xi > 0\}$) such that for all nonnegative integer l ,

$$\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l}{d\xi^l} \left(\frac{c_k}{\xi^k} \right) \quad (3.4)$$

as $|\xi| \rightarrow +\infty, \text{Im}\xi \leq 0$ ($\text{Im}\xi \geq 0$).

Let H' be the space of all polynomials and $H^- = H_0^- \oplus H'$; $H = H^+ \oplus H^-$. Denote by π^+ (π^-) respectively the projection on H^+ (H^-). For calculations, we take $H = \tilde{H} = \{\text{rational functions having no poles on the real axis}\}$ (\tilde{H} is a dense set in the topology of H). Then on \tilde{H} ,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \rightarrow 0^-} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi, \quad (3.5)$$

where Γ^+ is a Jordan close curve included $\text{Im}\xi > 0$ surrounding all the singularities of h in the upper half-plane and $\xi_0 \in \mathbf{R}$. Similarly, define π' on \tilde{H} ,

$$\pi' h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi) d\xi. \quad (3.6)$$

So, $\pi'(H^-) = 0$. For $h \in H \cap L^1(R)$, $\pi' h = \frac{1}{2\pi} \int_R h(v) dv$ and for $h \in H^+ \cap L^1(R)$, $\pi' h = 0$. Denote by \mathcal{B} Boutet de Monvel's algebra (for details, see [18] p.735), now we recall the main theorem in [19] p.29.

Theorem 3.3. (Fedosov-Golse-Leichtnam-Schrohe) *Let X and ∂X be connected, $\dim X = n \geq 3$, $A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathcal{B}$, and denote by p , b and s the local symbols of P , G and S respectively. Define:*

$$\begin{aligned} \widetilde{\text{Wres}}(A) &= \int_X \int_{\mathbf{S}} \text{tr}_E [p_{-n}(x, \xi)] \sigma(\xi) dx \\ &\quad + 2\pi \int_{\partial X} \int_{\mathbf{S}'} \{ \text{tr}_E [(\text{tr} b_{-n})(x', \xi')] + \text{tr}_F [s_{1-n}(x', \xi')] \} \sigma(\xi') dx', \end{aligned} \quad (3.7)$$

Then a) $\widetilde{\text{Wres}}([A, B]) = 0$, for any $A, B \in \mathcal{B}$; b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

Recall the definition of the Dirac operator D in [20]. Denote by $\sigma_l(A)$ the l -order symbol of an operator A . In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\tilde{e}_1, \dots, \tilde{e}_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\tilde{\nabla}(\tilde{e}_1, \dots, \tilde{e}_n) = (\tilde{e}_1, \dots, \tilde{e}_n)(\omega_{s,t}). \quad (3.8)$$

The Dirac operator

$$D = \sum_{i=1}^n c(\tilde{e}_i) [\tilde{e}_i - \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t)], \quad (3.9)$$

where $c(\tilde{e}_i)$ denotes the Clifford action. Then

$$\begin{aligned} D_T &= \sum_{i=1}^n c(\tilde{e}_i) [\tilde{e}_i - \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t)] + \frac{1}{4} \sum_{i \neq s \neq t} A_{ist} c(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t) \\ &\quad + \frac{1}{4} \sum_{i,s,t} [-A_{iit} c(\tilde{e}_t) + A_{isi} c(\tilde{e}_s) - A_{iss} c(\tilde{e}_i) + 2A_{iii} c(\tilde{e}_i)], \end{aligned} \quad (3.10)$$

$$\begin{aligned} D_T^* &= \sum_{i=1}^n c(\tilde{e}_i) [\tilde{e}_i - \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t)] + \frac{1}{4} \sum_{i \neq s \neq t} A_{ist} c(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t) \\ &\quad - \frac{1}{4} \sum_{i,s,t} [-A_{iit} c(\tilde{e}_t) + A_{isi} c(\tilde{e}_s) - A_{iss} c(\tilde{e}_i) + 2A_{iii} c(\tilde{e}_i)], \end{aligned} \quad (3.11)$$

and

$$\sigma_1(D_T) = \sigma_1(D_T^*) = \sqrt{-1}c(\xi); \quad (3.12)$$

$$\begin{aligned} \sigma_0(D_T) &= -\frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t) + \frac{1}{4} \sum_{i \neq s \neq t} A_{ist} c(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t) \\ &\quad + \frac{1}{4} \sum_{i,s,t} [-A_{iit} c(\tilde{e}_t) + A_{isi} c(\tilde{e}_s) - A_{iss} c(\tilde{e}_i) + 2A_{iii} c(\tilde{e}_i)], \end{aligned} \quad (3.13)$$

$$\begin{aligned} \sigma_0(D_T^*) &= -\frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t) + \frac{1}{4} \sum_{i \neq s \neq t} A_{ist} c(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t) \\ &\quad - \frac{1}{4} \sum_{i,s,t} [-A_{iit} c(\tilde{e}_t) + A_{isi} c(\tilde{e}_s) - A_{iss} c(\tilde{e}_i) + 2A_{iii} c(\tilde{e}_i)]. \end{aligned} \quad (3.14)$$

Hence by Lemma 2.1 in [10], we have

Lemma 3.4. *The symbol of the Dirac operator*

$$\sigma_{-1}(D_T^{-1}) = \sigma_{-1}((D_T^*)^{-1}) = \frac{\sqrt{-1}c(\xi)}{|\xi|^2}; \quad (3.15)$$

$$\sigma_{-2}(D_T^{-1}) = \frac{c(\xi)\sigma_0(D_T)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(\mathbf{d}x_j) \left[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right]; \quad (3.16)$$

$$\sigma_{-2}((D_T^*)^{-1}) = \frac{c(\xi)\sigma_0(D_T^*)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(\mathbf{d}x_j) \left[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right]. \quad (3.17)$$

Since Φ is a global form on ∂M , so for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates U of x_0 in ∂M (not in M) and compute $\Phi(x_0)$ in the coordinates $\tilde{U} = U \times [0, 1)$ and the metric $\frac{1}{h(x_n)}g^{\partial M} + \mathbf{d}x_n^2$. The dual metric of $g^{\partial M}$ on \tilde{U} is $\frac{1}{h(x_n)}g^{\partial M} + \mathbf{d}x_n^2$. Write $g_{ij}^M = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$; $g_M^{ij} = g^M(\mathbf{d}x_i, \mathbf{d}x_j)$, then

$$[g_{i,j}^M] = \begin{bmatrix} \frac{1}{h(x_n)}[g_{i,j}^{\partial M}] & 0 \\ 0 & 1 \end{bmatrix}; \quad [g_M^{i,j}] = \begin{bmatrix} h(x_n)[g_{\partial M}^{i,j}] & 0 \\ 0 & 1 \end{bmatrix}, \quad (3.18)$$

and

$$\partial_{x_s} g_{ij}^{\partial M}(x_0) = 0, \quad 1 \leq i, j \leq n-1; \quad g_{i,j}^M(x_0) = \delta_{ij}. \quad (3.19)$$

Let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal frame field in U about $g^{\partial M}$ which is parallel along geodesics and $e_i = \frac{\partial}{\partial x_i}(x_0)$, then $\{\tilde{e}_1 = \sqrt{h(x_n)}e_1, \dots, \tilde{e}_{n-1} = \sqrt{h(x_n)}e_{n-1}, \tilde{e}_n = \mathbf{d}x_n\}$ is the orthonormal frame field in \tilde{U} about g^M . Locally $S(TM)|_{\tilde{U}} \cong \tilde{U} \times \wedge_C^*(\frac{n}{2})$. Let $\{f_1, \dots, f_n\}$ be the orthonormal basis of $\wedge_C^*(\frac{n}{2})$. Take a spin frame field $\sigma : \tilde{U} \rightarrow Spin(M)$ such that $\pi\sigma = \{\tilde{e}_1, \dots, \tilde{e}_n\}$ where $\pi : Spin(M) \rightarrow O(M)$ is a double covering, then $\{[\sigma, f_i], 1 \leq i \leq 4\}$ is an orthonormal frame of $S(TM)|_{\tilde{U}}$. In the following, since the global form Φ is independent of the choice of the local frame, so we can compute $\mathbf{tr}_{S(TM)}$ in the frame $\{[\sigma, f_i], 1 \leq i \leq 4\}$. Let $\{E_1, \dots, E_n\}$ be the canonical basis of R^n and $c(E_i) \in cl_C(n) \cong Hom(\wedge_C^*(\frac{n}{2}), \wedge_C^*(\frac{n}{2}))$ be the Clifford action. By [20], then

$$c(\tilde{e}_i) = [(\sigma, c(E_i))]; \quad c(\tilde{e}_i)[(\sigma, f_i)] = [\sigma, (c(E_i)f_i)]; \quad \frac{\partial}{\partial x_i} = [(\sigma, \frac{\partial}{\partial x_i})], \quad (3.20)$$

then we have $\frac{\partial}{\partial x_i}c(\tilde{e}_i) = 0$ in the above frame. By Lemma 2.2 in [10], we have

Lemma 3.5. *With the metric $\frac{1}{h(x_n)}g^{\partial M} + \mathbf{d}x_n^2$ on M near the boundary*

$$\begin{aligned} \partial_{x_j}(|\xi|_{g^M}^2)(x_0) &= 0, \quad \text{if } j < n; \quad = h'(0)|\xi'|_{g^{\partial M}}^2, \quad \text{if } j = n. \\ \partial_{x_j}(c(\xi))(x_0) &= 0, \quad \text{if } j < n; \quad = \partial_{x_n}(c(\xi'))(x_0), \quad \text{if } j = n. \end{aligned} \quad (3.21)$$

where $\xi = \xi' + \xi_n \mathbf{d}x_n$

Then an application of Lemma 2.3 in [10] shows

Lemma 3.6.

$$\sigma_1(D_T) = \sigma_1(D_T^*) = \sqrt{-1}c(\xi); \quad (3.22)$$

$$\begin{aligned} \sigma_0(D_T) &= -\frac{3}{4}h'(0)c(dx_n) + \frac{1}{4} \sum_{i \neq s \neq t} A_{ist}c(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t) \\ &\quad + \frac{1}{4} \sum_{i,s,t} [-A_{iit}c(\tilde{e}_t) + A_{isi}c(\tilde{e}_s) - A_{iss}c(\tilde{e}_i) + 2A_{iii}c(\tilde{e}_i)], \end{aligned} \quad (3.23)$$

$$\begin{aligned} \sigma_0(D_T^*) &= -\frac{3}{4}h'(0)c(dx_n) + \frac{1}{4} \sum_{i \neq s \neq t} A_{ist}c(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t) \\ &\quad - \frac{1}{4} \sum_{i,s,t} [-A_{iit}c(\tilde{e}_t) + A_{isi}c(\tilde{e}_s) - A_{iss}c(\tilde{e}_i) + 2A_{iii}c(\tilde{e}_i)]. \end{aligned} \quad (3.24)$$

Now we can compute Φ (see formula (2.13) for definition of Φ), since the sum is taken over $-r - \ell + k + j + |\alpha| = 3$, $r, \ell \leq -1$, then we have the following five cases:

Case a(I): $r = -1$, $\ell = -1$, $k = j = 0$, $|\alpha| = 1$

From (2.13), we have

$$\text{Case a(I)} = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi'}^{\alpha} \pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{x'}^{\alpha} \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (3.25)$$

By Lemma 3.5, for $j < n$

$$\partial_{x_i} \sigma_{-1}(D_T^{-1})(x_0) = \partial_{x_i} \left(\frac{\sqrt{-1}c(\xi)}{|\xi|^2} \right) (x_0) = \frac{\sqrt{-1} \partial_{x_i} [c(\xi)](x_0)}{|\xi|^2} - \frac{\sqrt{-1}c(\xi) \partial_{x_i} (|\xi|^2)(x_0)}{|\xi|^4} = 0, \quad (3.26)$$

so Case a(I) vanishes.

Case a(II): $r = -1$, $\ell = -1$, $k = |\alpha| = 0$, $j = 1$

From (2.13), we have

$$\text{Case a(II)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{\xi_n}^2 \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (3.27)$$

Similarly to (2.2.18) in [10], we have

$$\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1})(x_0)|_{|\xi'|=1} = \frac{\partial_{x_n} [c(\xi')](x_0)}{2(\xi_n - i)} + \sqrt{-1}h'(0) \left[\frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(dx_n)}{4(\xi_n - i)^2} \right]; \quad (3.28)$$

$$\partial_{\xi_n}^2 \sigma_{-1}(D_T^{-1}) = \sqrt{-1} \left(-\frac{6\xi_n c(dx_n) + 2c(\xi')}{|\xi|^4} + \frac{8\xi_n^2 c(\xi)}{|\xi|^6} \right). \quad (3.29)$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then

$$\begin{aligned} \text{tr}[c(\xi')c(dx_n)] &= 0; \quad \text{tr}[c(dx_n)^2] = -4; \quad \text{tr}[c(\xi')^2](x_0)|_{|\xi'|=1} = -4; \\ \text{tr}[\partial_{x_n} [c(\xi')]c(dx_n)] &= 0; \quad \text{tr}[\partial_{x_n} c(\xi') \times c(\xi')](x_0)|_{|\xi'|=1} = -2h'(0). \end{aligned} \quad (3.30)$$

For more trace expansions, we can see [21]. Hence we conclude that

$$\text{trace}[\partial_{\xi}^{\alpha} \pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{x'}^{\alpha} \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) = \frac{2ih'(0)}{(\xi_n - i)^2(\xi_n + i)^3}. \quad (3.31)$$

Therefore

$$\text{Case a(II)} = -\frac{3}{8}\pi h'(0)\Omega_3 dx', \quad (3.32)$$

where Ω_3 is the canonical volume of S^3 .

Case a(III): $r = -1$, $\ell = -1$, $j = |\alpha| = 0$, $k = 1$

From (2.13), we have

$$\text{Case a(III)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (3.33)$$

Similarly to (2.2.27) in [10], we have

$$\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}((D^*)^{-1})(x_0)|_{|\xi'|=1} = -\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}, \quad (3.34)$$

and

$$\partial_{\xi_n} \partial_{x_n} \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = -\sqrt{-1}h'(0) \left[\frac{c(dx_n)}{|\xi|^4} - 4\xi_n \frac{c(\xi') + \xi_n c(dx_n)}{|\xi|^6} \right] - \frac{2\sqrt{-1}\xi_n \partial_{x_n} c(\xi')(x_0)}{|\xi|^4}. \quad (3.35)$$

Combining (3.34) and (3.35), we obtain

$$\text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D_T^{-1}) \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(D_T^{-1})](x_0) = \frac{2h'(0)(i - 2\xi_n - i\xi_n^2)}{(\xi_n - i)^4(\xi_n + i)^3}. \quad (3.36)$$

Then

$$\text{Case a(III)} = \frac{3}{8}\pi h'(0)\Omega_3 dx', \quad (3.37)$$

where Ω_3 is the canonical volume of S^3 . Thus the sum of Case a(II) and Case a(III) is zero.

Case b: $r = -2$, $\ell = -1$, $k = j = |\alpha| = 0$

By (2.13), we get

$$\text{Case b} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-1}) \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (3.38)$$

Then an application of Lemma 3.4 and Lemma 3.5 shows

$$\begin{aligned} \sigma_{-2}((D_T^*)^{-1})(x_0) &= \frac{c(\xi)\sigma_0(D_T^*)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right](x_0) \\ &= \frac{c(\xi)\sigma_0(D_T^*)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[\partial_{x_n}(c(\xi'))(x_0) - c(\xi)h'(0)|\xi'|_{g^{\partial M}}^2 \right]. \end{aligned} \quad (3.39)$$

Hence in this case,

$$\pi_{\xi_n}^+ \sigma_{-2}((D^*)^{-1})(x_0) := A_1 + A_2, \quad (3.40)$$

where

$$\begin{aligned} A_1 &= -\frac{h'(0)}{2} \left[\frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} [ic(\xi') - c(dx_n)] \right] \\ &\quad + \frac{-1}{4(\xi_n - i)^2} \left[(2 + i\xi_n)c(\xi')\alpha_0 c(\xi') + i\xi_n c(dx_n)\alpha_0 c(dx_n) + (2 + i\xi_n)c(\xi')c(dx_n)\partial_{x_n} c(\xi') \right. \\ &\quad \left. + ic(dx_n)\alpha_0 c(\xi') + ic(\xi')\alpha_0 c(dx_n) - i\partial_{x_n} c(\xi') \right]; \end{aligned} \quad (3.41)$$

$$\begin{aligned} A_2 &= \frac{-1}{4(\xi_n - i)^2} \left[(2 + i\xi_n)c(\xi')\beta_0 c(\xi') + i\xi_n c(dx_n)\beta_0 c(dx_n) + ic(dx_n)\beta_0 c(\xi') \right. \\ &\quad \left. + ic(\xi')\beta_0 c(dx_n) \right] \end{aligned} \quad (3.42)$$

and

$$\alpha_0 = -\frac{3}{4}h'(0)c(dx_n), \quad (3.43)$$

$$\beta_0 = \frac{1}{4} \sum_{i \neq s \neq t} A_{ist} c(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t) - \frac{1}{4} \sum_{i,s,t} \left[-A_{iit} c(\tilde{e}_t) + A_{isi} c(\tilde{e}_s) - A_{iss} c(\tilde{e}_i) + 2A_{iii} c(\tilde{e}_i) \right]. \quad (3.44)$$

On the other hand,

$$\partial_{\xi_n} \sigma_{-1}(D_T^{-1}) = \frac{-2i\xi_n c(\xi')}{(1+\xi_n^2)^2} + \frac{i(1-\xi_n^2)c(\mathbf{d}x_n)}{(1+\xi_n^2)^2}. \quad (3.45)$$

From (2.2.39), (2.2.41) and (2.2.42) in [10], we have

$$-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[A_1 \times \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' = \frac{9}{8} \pi h'(0) \Omega_3 \mathbf{d}x'. \quad (3.46)$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities

$$\text{tr}[c(\tilde{e}_i)c(\mathbf{d}x_n)] = 0, i < n; \quad \text{tr}[c(\tilde{e}_i)c(\mathbf{d}x_n)] = -4, i = n. \quad (3.47)$$

Combining (3.42), (3.45) and (3.47), we obtain

$$\text{trace}[A_2 \times \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) = \frac{ic_0}{2(\xi_n + i)^2(\xi_n - i)}, \quad (3.48)$$

where

$$c_0 = -2 \sum_i A_{iin}. \quad (3.49)$$

Hence from (3.42) and (3.45), we have

$$\begin{aligned} & -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[A_2 \times \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' \\ &= -i \Omega_3 \int_{\Gamma^+} \frac{ic_0}{2(\xi_n + i)^2(\xi_n - i)} \mathbf{d}\xi_n \mathbf{d}x' \\ &= -i 2\pi i \Omega_3 \left[\frac{ic_0}{2(\xi_n + i)^2} \right]^{(1)}|_{\xi_n=i} \mathbf{d}x' \\ &= \frac{1}{4} \pi c_0 \Omega_3 \mathbf{d}x'. \end{aligned} \quad (3.50)$$

Combining (3.46) and (3.50), we have

$$\text{case } b = \left[\frac{9}{8} h'(0) - \frac{1}{2} \sum_i A_{iin} \right] \pi \Omega_3 \mathbf{d}x'. \quad (3.51)$$

Case c: $r = -1, \ell = -2, k = j = |\alpha| = 0$

From (2.13), we have

$$\text{case } c = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{\xi_n} \sigma_{-2}(D_T^{-1})](x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x'. \quad (3.52)$$

Then an application of Lemma 3.4 shows

$$\pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) = \frac{c(\xi') + ic(\mathbf{d}x_n)}{2(\xi_n - i)}. \quad (3.53)$$

By Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned}\sigma_{-2}((D_T)^{-1})(x_0) &= \frac{c(\xi)\sigma_0(D_T)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right](x_0) \\ &= \frac{c(\xi)\sigma_0(D_T)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[\partial_{x_n}(c(\xi'))(x_0) - c(\xi)h'_{x_n}(0)|\xi'|_{g^{\partial M}}^2 \right].\end{aligned}\quad (3.54)$$

Hence in this case,

$$\partial_{\xi_n}\sigma_{-2}(D_T^{-1})(x_0) := B_1 + B_2, \quad (3.55)$$

where

$$\begin{aligned}B_1 &= \frac{1}{(1+\xi_n^2)^3} \left[(2\xi_n - 2\xi_n^3)c(dx_n)\alpha_0c(dx_n) + (1 - 3\xi_n^2)c(dx_n)\alpha_0c(\xi') \right. \\ &\quad + (1 - 3\xi_n^2)c(\xi')\alpha_0c(dx_n) - 4\xi_nc(\xi')\alpha_0c(\xi') + (3\xi_n^2 - 1)\partial_{x_n}c(\xi') - 4\xi_nc(\xi')c(dx_n)\partial_{x_n}c(\xi') \\ &\quad \left. + 2h'(0)c(\xi') + 2h'(0)\xi_nc(dx_n) \right] + 6\xi_nh'(0)\frac{c(\xi)c(dx_n)c(\xi')}{(1+\xi_n^2)^4};\end{aligned}\quad (3.56)$$

$$\begin{aligned}B_2 &= \frac{1}{(1+\xi_n^2)^3} \left[(2\xi_n - 2\xi_n^3)c(dx_n)\beta_1c(dx_n) + (1 - 3\xi_n^2)c(dx_n)\beta_1c(\xi') \right. \\ &\quad \left. + (1 - 3\xi_n^2)c(\xi')\beta_1c(dx_n) - 4\xi_nc(\xi')\beta_1c(\xi') \right]\end{aligned}\quad (3.57)$$

and

$$\beta_1 = \frac{1}{4} \sum_{i \neq s \neq t} A_{ist}c(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t) + \frac{1}{4} \sum_{i,s,t} [-A_{iit}c(\tilde{e}_t) + A_{isi}c(\tilde{e}_s) - A_{iss}c(\tilde{e}_i) + 2A_{iii}c(\tilde{e}_i)]. \quad (3.58)$$

Then similarly to computations of the (3.50), we have

$$-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \times B_1](x_0) d\xi_n \sigma(\xi') dx' = -\frac{9}{8} \pi h'(0) \Omega_3 dx'. \quad (3.59)$$

From (3.53) and (3.57) we obtain

$$\text{trace}[\pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \times B_2](x_0) = \frac{-i\tilde{c}_0}{(\xi_n + i)^3(\xi_n - i)}, \quad (3.60)$$

where

$$\tilde{c}_0 = 2 \sum_i A_{iin}. \quad (3.61)$$

Then

$$\begin{aligned}& -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \times B_2](x_0) d\xi_n \sigma(\xi') dx' \\ &= -i\Omega_3 \int_{\Gamma^+} \frac{i\tilde{c}_0}{(\xi_n + i)^3(\xi_n - i)} d\xi_n dx' \\ &= 2\pi i\Omega_3 \frac{\tilde{c}_0}{(\xi_n + i)^3} \Big|_{\xi_n=i} dx' \\ &= -\frac{1}{4} \pi \tilde{c}_0 \Omega_3 dx'.\end{aligned}\quad (3.62)$$

Combining (3.59) and (3.62), we have

$$\text{case } c = \left[-\frac{9}{8} h'(0) - \frac{1}{2} \sum_i A_{iin} \right] \pi \Omega_3 dx'. \quad (3.63)$$

Now Φ is the sum of the **case (a, b, c)**, so

$$\sum \text{case a, b, c} = - \sum_i A_{iin} \pi \Omega_3 dx'. \quad (3.64)$$

Hence we conclude that

Theorem 3.7. *Let M be a 4-dimensional compact manifold with the boundary ∂M and $\tilde{\nabla}$ be an orthogonal connection with torsion. Then we get the volumes associated to D^*D ,*

$$Vol_4^{(1,1)} = -\frac{1}{48\pi^2} \int_M \tilde{R}(x) dx - \int_{\partial M} \sum_i A_{iin} \pi \Omega_3 dx', \quad (3.65)$$

where $\tilde{R} = R + 18\text{div}(V) - 54|V|^2 - 9 \|T\|^2$ and $\int_M \text{div}(V) dVol_M = - \int_{\partial M} g(n, V) dVol_{\partial M}$.

4. The gravitational action for 4-dimensional manifolds with boundary

Firstly, we recall the Einstein-Hilbert action with torsion for manifolds with boundary (see [10] or [11]),

$$I_{Gr} = \frac{1}{16\pi} \int_M \tilde{R} dvol_M + 2 \int_{\partial M} \tilde{K} dvol_{\partial M} := I_{Gr,i} + I_{Gr,b}, \quad (4.1)$$

where $\tilde{R} = R + 18\text{div}(V) - 54|V|^2 - 9 \|T\|^2$ be the scalar curvature of this orthogonal connection without the Cartan type torsion S . And

$$\tilde{K} = K + \sum_i A_{iin}; \quad K = \sum_{1 \leq i,j \leq n-1} K_{i,j} g_{\partial M}^{i,j}, \quad (4.2)$$

where $K_{i,j}$ is the second fundamental form, or extrinsic curvature. Take the metric in Section 2, and by Lemma A.2 in [10], for $n = 4$, we assume the manifold approach the boundary ∂_M is flat, then

$$\tilde{K}(x_0) = \sum_i A_{iin}, \quad K(x_0) = 0. \quad (4.3)$$

Let

$$\widetilde{\text{Wres}}[\pi^+(D_T^*)^{-1} \circ \pi^+ D_T^{-1}] = \widetilde{\text{Wres}}_i[\pi^+(D_T^*)^{-1} \circ \pi^+ D_T^{-1}] + \widetilde{\text{Wres}}_b[\pi^+(D_T^*)^{-1} \circ \pi^+ D_T^{-1}], \quad (4.4)$$

where

$$\widetilde{\text{Wres}}_i[\pi^+(D_T^*)^{-1} \circ \pi^+ D_T^{-1}] = \int_M \int_{|\xi|=1} \text{trace}_{S(TM)}[\sigma_{-4}((D_T^*)^{-1} \circ D_T^{-1})] \sigma(\xi) dx \quad (4.5)$$

and

$$\begin{aligned} & \widetilde{\text{Wres}}_b[\pi^+((D_T^*)^{-1} \circ \pi^+ D_T^{-1})] \\ &= \int_{\partial M} \int_{|\xi|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{S(TM)}[\partial_{x_n}^j \partial_{\xi}^{\alpha} \partial_{\xi_n}^k \sigma_r^+(((D_T^*)^{-1})(x', 0, \xi', \xi_n) \\ & \quad \times \partial_{x'}^{\alpha} \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(D_T^{-1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx' \end{aligned} \quad (4.6)$$

denote the interior term and boundary term of $\widetilde{\text{Wres}}[\pi^+(D_T^*)^{-1} \circ \pi^+ D_T^{-1}]$.

Combining (3.65), (4.1) and (4.4), we obtain

Theorem 4.1. *Let M be a 4-dimensional compact manifold with the boundary ∂M and $\tilde{\nabla}$ be an orthogonal connection with torsion. Then we get the volumes associated to D^*D ,*

$$\begin{aligned} I_{Gr,i} &= -3\pi \widetilde{\text{Wres}}_i[\pi^+(D_T^*)^{-1} \circ \pi^+ D_T^{-1}]; \\ I_{Gr,b} &= \frac{-2}{\pi \Omega_3} \widetilde{\text{Wres}}_b[\pi^+(D_T^*)^{-1} \circ \pi^+ D_T^{-1}]. \end{aligned} \quad (4.7)$$

5. A Kastler-Kalau-Walze type theorem for 6-dimensional spin manifolds with boundary associated to $(D_T^*)^2$ and D_T^2

In this section, We compute the lower dimensional volume $\text{Vol}_6^{(2,2)}$ for 6-dimensional spin manifolds with boundary of metric $g^M = \frac{1}{h(x_n)}g^{\partial M} + dx_n^2$ and get a Kastler-Kalau-Walze type theorem in this case.

Firstly, we compute $\int_{\partial M} \Phi$ in this case. By Lemma 1 in [11], we have

Lemma 5.1.

$$\sigma_{-2}((D_T^*)^{-2}) = |\xi|^{-2}; \quad (5.1)$$

$$\sigma_{-2}(D_T^{-2}) = |\xi|^{-2}; \quad (5.2)$$

$$\sigma_{-3}((D_T^*)^{-2}) = -\sqrt{-1}|\xi|^{-4}\xi_k(\tilde{\Gamma}^k - 2\hat{\delta}^k) - \sqrt{-1}|\xi|^{-6}2\xi^j\xi_\alpha\xi_\beta\partial_jg^{\alpha\beta} - 2\sqrt{-1}|\xi|^{-4}(u-v)c(\xi); \quad (5.3)$$

$$\sigma_{-3}(D_T^{-2}) = -\sqrt{-1}|\xi|^{-4}\xi_k(\tilde{\Gamma}^k - 2\check{\delta}^k) - \sqrt{-1}|\xi|^{-6}2\xi^j\xi_\alpha\xi_\beta\partial_jg^{\alpha\beta} - 2\sqrt{-1}|\xi|^{-4}(u+v)c(\xi). \quad (5.4)$$

Now we can compute Φ (see formula (2.13) for the definition of Φ), since the sum is taken over $-r-l+k+j+|\alpha|=5$, $r, l \leq -2$, then we have the following five cases:

case â) I) $r = -2, l = -2, k = j = 0, |\alpha| = 1$

From (2.13) we have

$$\text{case â) I) } = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-2}) \times \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-2}(D_T^{-2})](x_0) d\xi_n \sigma(\xi') dx'. \quad (5.5)$$

By Lemma 3.5, for $i < n$, then

$$\partial_{x_i} \sigma_{-2}(D_T^{-2})(x_0) = \partial_{x_i} \left(\frac{1}{|\xi|^2} \right) (x_0) = - \frac{\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0.$$

Then case â) I) vanishes.

case â) II) $r = -1, l = -1, k = |\alpha| = 0, j = 1$

From (2.13) we have

$$\text{case â) II) } = - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-2}) \times \partial_{\xi_n}^2 \sigma_{-2}(D_T^{-2})](x_0) d\xi_n \sigma(\xi') dx'. \quad (5.6)$$

By Lemma 3.5 and Lemma 5.1, we have

$$\partial_{\xi_n}^2 \sigma_{-2}(D_T^{-2})(x_0) = \partial_{\xi_n}^2 \left(\frac{1}{|\xi|^2} \right) (x_0) = \frac{-2 + 6\xi_n^2}{(1 + \xi_n^2)^3}, \quad (5.7)$$

and

$$\partial_{x_n} \sigma_{-2}((D_T^*)^{-2})(x_0) = \frac{-h'(0)}{(1 + \xi_n^2)^3}. \quad (5.8)$$

Then

$$\pi_{\xi_n}^+ [\partial_{x_n} \sigma_{-2}((D_T^*)^{-2})](x_0)|_{|\xi'|=1} = \frac{(i\xi_n^2 + 2)h'(0)}{4(\xi_n - i)^2}. \quad (5.9)$$

Combining (5.7) and (5.9), we obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{(i\xi_n^2 + 2)h'(0)}{4(\xi_n - i)^2} \times \frac{-2 + 6\xi_n^2}{(1 + \xi_n^2)^3} d\xi_n \\ &= -\frac{1}{2} \int_{\Gamma^+} \frac{(3\xi_n^2 - 1)(-2h'(0) - i\xi_n h'(0))}{(\xi_n - i)^5 (\xi_n + i)^3} d\xi_n dx' \\ &= -\frac{1}{2} \pi i \left[\frac{(3\xi_n^2 - 1)(-2h'(0) - i\xi_n h'(0))}{(\xi_n + i)^3} \right]^{(4)} \Big|_{\xi_n = i} dx' \\ &= \frac{5h'(0)}{32}. \end{aligned} \quad (5.10)$$

Since $n = 6$, $\text{tr}_{S(TM)}[\text{id}] = \dim(\wedge^*(3)) = 8$. Combining (5.6) and (5.10), we have

$$\text{case } \hat{a}) \text{ II}) = -\frac{5h'(0)}{8}\Omega_4 dx'. \quad (5.11)$$

where Ω_4 is the canonical volume of S^4 .

case } \hat{a}) \text{ III}) $r = -2$, $l = -2$, $j = |\alpha| = 0$, $k = 1$

From (2.13) and an integration by parts, we get

$$\begin{aligned} \text{case } \hat{a}) \text{ III}) &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-2}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-2}(D_T^{-2})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-2}) \times \partial_{x_n} \sigma_{-2}(D_T^{-2})](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \quad (5.12)$$

By Lemma 3.5 and Lemma 5.1, we have

$$\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-2})(x_0)|_{|\xi'|=1} = \frac{-i}{(\xi_n - i)^3}. \quad (5.13)$$

Substituting (5.8) and (5.13) into (5.12), one sees that

$$\begin{aligned} \text{case } \hat{a}) \text{ III}) &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{8ih'(0)}{(\xi_n - i)^5 (\xi_n + i)^2} d\xi_n \sigma(\xi') dx' \\ &= \frac{5h'(0)}{8} \pi \Omega_4 dx'. \end{aligned} \quad (5.14)$$

case } \hat{b}) $r = -2$, $l = -3$, $k = j = |\alpha| = 0$

From (2.13) and an integration by parts, we get

$$\begin{aligned} \text{case } \hat{b}) &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-2}) \times \partial_{\xi_n} \sigma_{-3}(D_T^{-2})](x_0) d\xi_n \sigma(\xi') dx' \\ &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-2}) \times \sigma_{-3}(D_T^{-2})](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \quad (5.15)$$

By Lemma 5.1, we have

$$\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-2})(x_0)|_{|\xi'|=1} = \frac{i}{2(\xi_n - i)^2}. \quad (5.16)$$

In the normal coordinate, $g^{ij}(x_0) = \delta_i^j$ and $\partial_{x_j}(g^{\alpha\beta})(x_0) = 0$, if $j < n$; $= h'(0)\delta_\beta^\alpha$, if $j = n$. So by Lemma A.2 in [10], we have $\Gamma^n(x_0) = \frac{5}{2}h'(0)$ and $\Gamma^k(x_0) = 0$ for $k < n$. By the definition of δ^k and Lemma 2.3 in [10], we have $\delta^n(x_0) = 0$ and $\delta^k = \frac{1}{4}h'(0)c(\tilde{e}_k)c(\tilde{e}_n)$ for $k < n$. So

$$\begin{aligned} &\sigma_{-3}(D^{-2})(x_0)|_{|\xi'|=1} \\ &= -\sqrt{-1}|\xi|^{-4}\xi_k(\tilde{\Gamma}^k - 2\delta^k)(x_0)|_{|\xi'|=1} - \sqrt{-1}|\xi|^{-6}2\xi^j\xi_\alpha\xi_\beta\partial_j g^{\alpha\beta}(x_0)|_{|\xi'|=1} - 2\sqrt{-1}|\xi|^{-4}(u+v)c(\xi) \\ &= \frac{-i}{(1+\xi_n^2)^2} \left(-\frac{1}{2}h'(0) \sum_{k<n} \xi_k c(\tilde{e}_k)c(\tilde{e}_n) + \xi_n \frac{5}{2}h'(0) \right) - \frac{2i\xi_n h'(0)}{(1+\xi_n^2)^3} - 2\sqrt{-1}|\xi|^{-4}(u+v)c(\xi). \end{aligned} \quad (5.17)$$

We note that $\int_{|\xi'|=1} \xi_1 \cdots \xi_{2q+1} \sigma(\xi') = 0$ and $\text{tr}[v \times c(dx_n)] = 4 \sum_i A_{iin}$. Then

$$\begin{aligned}
\text{case } \hat{\mathbf{b}}) &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-2}) \times \sigma_{-3}(D_T^{-2})](x_0) d\xi_n \sigma(\xi') dx' \\
&= -\frac{15h'(0)}{8} \pi \Omega_4 dx' + i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}\left[\frac{i}{2(\xi_n - i)^2} \times \left(-2\sqrt{-1}|\xi|^{-4}(u+v)c(\xi)\right)\right](x_0) d\xi_n \sigma(\xi') dx' \\
&= -\frac{15h'(0)}{8} \pi \Omega_4 dx' + i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{1}{(\xi_n - i)^2(1 + \xi_n^2)^2} \text{tr}\left[(u+v)c(\xi)\right](x_0) d\xi_n \sigma(\xi') dx' \\
&= -\frac{15h'(0)}{8} \pi \Omega_4 dx' + 4i \sum_i A_{iin} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{\xi_n}{(\xi_n - i)^2(1 + \xi_n^2)^2} d\xi_n \sigma(\xi') dx' \\
&= \left(-\frac{15h'(0)}{8} - \frac{1}{2} \sum_i A_{iin}\right) \pi \Omega_4 dx'. \tag{5.18}
\end{aligned}$$

case $\hat{\mathbf{c}}$ $r = -3, l = -2, k = j = |\alpha| = 0$

From (2.13) we have

$$\text{case } \hat{\mathbf{c}}) = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-3}((D_T^*)^{-2}) \times \partial_{\xi_n} \sigma_{-2}(D_T^{-2})](x_0) d\xi_n \sigma(\xi') dx'. \tag{5.19}$$

By (24) in [11], we have

$$\text{case } \hat{\mathbf{c}}) = \text{case } \hat{\mathbf{b}}) - i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_{-2}(D_T^{-2}) \times \sigma_{-3}((D_T^*)^{-2})] d\xi_n \sigma(\xi') dx'. \tag{5.20}$$

Then an application of Lemma 5.1 shows

$$\partial_{\xi_n} \sigma_{-2}(D_T^{-2})(x_0) = \frac{-2\xi_n}{(1 + \xi_n^2)^2}. \tag{5.21}$$

Combining (5.3) and (5.21), we obtain

$$\begin{aligned}
&-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_{-2}(D_T^{-2}) \times \sigma_{-3}((D_T^*)^{-2})] d\xi_n \sigma(\xi') dx' \\
&= \frac{15h'(0)}{4} \pi \Omega_4 dx' - i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}\left[\frac{-2\xi_n}{(1 + \xi_n^2)^2} \times \left(-2\sqrt{-1}|\xi|^{-4}(u-v)c(\xi)\right)\right](x_0) d\xi_n \sigma(\xi') dx' \\
&= \frac{15h'(0)}{4} \pi \Omega_4 dx' - i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{4i\xi_n}{(1 + \xi_n^2)^4} \text{tr}\left[(u-v)c(\xi)\right](x_0) d\xi_n \sigma(\xi') dx' \\
&= \frac{15h'(0)}{4} \pi \Omega_4 dx' - \sum_i A_{iin} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{16\xi_n^2}{(\xi_n - i)^4(\xi_n + i)^4} d\xi_n \sigma(\xi') dx' \\
&= \left(\frac{15h'(0)}{4} - \sum_i A_{iin}\right) \pi \Omega_4 dx'. \tag{5.22}
\end{aligned}$$

From (5.18) and (5.22), we have

$$\text{case } \hat{\mathbf{c}}) = \left(\frac{15h'(0)}{8} - \frac{3}{2} \sum_i A_{iin}\right) \pi \Omega_4 dx'. \tag{5.23}$$

Since Φ is the sum of the cases $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$, so $\Phi = -2 \sum_i A_{iin} \pi \Omega_4 dx'$. Hence we conclude that

Theorem 5.2. *Let M be a 6-dimensional compact spin manifold with the boundary ∂M and $\tilde{\nabla}$ be an orthogonal connection with torsion. Then we get the volumes associated to $D_T^* D_T$ with torsion on \widehat{M}*

$$\widetilde{Wres}(\pi^+(D_T^*)^{-2} \circ \pi^+(D_T)^{-2}) = -2 \int_{\partial M} \sum_i A_{iin} \pi \Omega_4 dx'; \quad (5.24)$$

when $V = 0$,

$$Wres(\pi^+(D_T^*)^{-2} \circ \pi^+(D_T)^{-2}) = -\frac{1}{48\pi^2} \int_M (R - 9 \|T\|^2)(x) dx. \quad (5.25)$$

6. The Kastler-Kalau-Walze theorem for 4-dimensional spin manifolds with boundary about Dirac operator $P^+ D_T^* D_T$

Next we consider the volume form ω^g acting on the spinor bundle ΣM . Setting $P = P^+ = \frac{1}{2}(id_\Sigma + \omega^g)$ we have a parallel field of orthogonal projections. If we now calculate the Seeley-deWitt coefficients of $H^+ = P^+ D^* D$, we obtain relations to Loop Quantum Gravity. The Holst term for the modified connection $\tilde{\nabla}$ is the 4-form

$$\tilde{C}_H = 18(dT - \langle T, *V \rangle \omega^g), \quad (6.1)$$

where $\omega^g = e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*$, see Remark 3.2 and Proposition 3.3 in [15].

Theorem 6.1. [15] *Let M be a 4-dimensional compact manifold without boundary and let \tilde{R} be the scalar curvature of the modified connection $\tilde{\nabla}$. Then we get the volumes associated to $P^+ D_T^* D_T$,*

$$Wres(P^+(D_T^* D_T)^{-1}) = -\frac{1}{96\pi^2} \int_M (\tilde{R}\omega^g + \tilde{C}_H), \quad (6.2)$$

where $\tilde{R} = R + 18\text{div}(V) - 54|V|^2 - 9 \|T\|^2$ be the scalar curvature of the modified connection $\tilde{\nabla}$.

Theorem 6.2. [15] *Let M be a 6-dimensional compact manifold without boundary and let \tilde{R} be the scalar curvature of the modified connection $\tilde{\nabla}$. Then we get the volumes associated to $P^+ D_T^* D_T$,*

$$\begin{aligned} Wres(P^+(D_T^* D_T)^{-1}) &= \frac{11}{1440} \mathcal{X}(M) - \frac{1}{96} p_1(M) - \frac{1}{640\pi^2} \int_M \|C^g\|^2 dx \\ &\quad - \frac{3}{64\pi^2} \int_M (\|\delta T\|^2 + \|d(V)\|^2) dx + \frac{1}{1152\pi^2} \int_M \tilde{R} \tilde{C}_H. \end{aligned} \quad (6.3)$$

where C^g denote the Weyl curvature of the Levi-Civita connection, and $p_1(M)$ denote the first Pontryagin class of M and $\mathcal{X}(M)$ denote the Euler characteristics of M .

Now we can compute Φ (see formula (2.13) for definition of Φ), since the sum is taken over $-r - \ell + k + j + |\alpha| = 3$, $r, \ell \leq -1$, then we have the following five cases:

Case \bar{a}) I): $r = -1, \ell = -1, k = j = 0, |\alpha| = 1$

By (2.13), we get

$$\begin{aligned} \text{Case } \bar{a}) \text{ I)} &= - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ P^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \omega^g \sigma_{-1}((D_T^*)^{-1}) \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' \end{aligned} \quad (6.4)$$

By Lemma 3.5, for $j < n$

$$\partial_{x_i} \sigma_{-1}(D_T^{-1})(x_0) = \partial_{x_i} \left(\frac{\sqrt{-1}c(\xi)}{|\xi|^2} \right) (x_0) = \frac{\sqrt{-1}\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2} - \frac{\sqrt{-1}c(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0, \quad (6.5)$$

so Case \bar{a}) I) vanishes.

Case \bar{a}) II): $r = -1$, $\ell = -1$, $k = |\alpha| = 0$, $j = 1$

From (2.13), we have

$$\begin{aligned} \text{Case } \bar{a}) \text{ II)} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ P^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{\xi_n}^2 \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{4} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{\xi_n}^2 \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad - \frac{1}{4} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \omega^g \sigma_{-1}((D_T^*)^{-1}) \partial_{\xi_n}^2 \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' \end{aligned} \quad (6.6)$$

Similarly to **Case a(II)** in [10], we have

$$-\frac{1}{4} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{\xi_n}^2 \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' = -\frac{3}{16} \pi h'(0) \Omega_3 dx'. \quad (6.7)$$

On the other hand, similarly to (2.2.18) in [10], we have

$$\begin{aligned} &\partial_{x_n} \pi_{\xi_n}^+ \omega^g \sigma_{-1}((D_T^*)^{-1})(x_0)|_{|\xi'|=1} \\ &= c(\tilde{e}_1) c(\tilde{e}_2) c(\tilde{e}_3) c(dx_n) \left[\frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n - i)} + \frac{(2i - \xi_n)h'(0)c(\xi')}{4(\xi_n - i)^2} - \frac{h'(0)c(dx_n)}{4(\xi_n - i)^2} \right] \end{aligned} \quad (6.8)$$

and

$$\begin{aligned} \partial_{\xi_n}^2 \sigma_{-1}(D_T^{-1}) &= \sqrt{-1} \left(-\frac{6\xi_n c(dx_n) + 2c(\xi')}{|\xi|^4} + \frac{8\xi_n^2 c(\xi)}{|\xi|^6} \right) \\ &= \frac{6i\xi_n^2 - 2i}{(1 + \xi_n^2)^3} c(\xi') + \frac{2i\xi_n^3 - 6i\xi_n}{(1 + \xi_n^2)^3} c(dx_n). \end{aligned} \quad (6.9)$$

By the relation of the Clifford action and $\text{tr} AB = \text{tr} BA$, considering for $i < n$, $\int_{|\xi'|=1} \{\xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2d+1}}\} \sigma(\xi') = 0$, then

$$\text{tr}[c(\tilde{e}_1) c(\tilde{e}_2) c(\tilde{e}_3) c(\xi')] = 0, \quad \text{tr}[c(\tilde{e}_1) c(\tilde{e}_2) c(\tilde{e}_3) c(dx_n) \partial_{x_n}[c(\xi')] c(\xi')] = 0. \quad (6.10)$$

Hence in this case,

$$\text{trace}[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \omega^g \sigma_{-1}((D_T^*)^{-1}) \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) = 0. \quad (6.11)$$

Therefore

$$\text{Case } \bar{a}) \text{ II)} = -\frac{3}{16} \pi h'(0) \Omega_3 dx', \quad (6.12)$$

where Ω_3 is the canonical volume of S^3 .

Case \bar{a}) III): $r = -1$, $\ell = -1$, $j = |\alpha| = 0$, $k = 1$

From (2.13), we have

$$\begin{aligned} \text{Case } \bar{a}) \text{ III)} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ P^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{4} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad - \frac{1}{4} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \omega^g \sigma_{-1}((D_T^*)^{-1}) \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \quad (6.13)$$

Similarly to **Case a(III)** in [10], we have

$$-\frac{1}{4} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}((D_T^*)^{-1}) \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' = \frac{3}{16} \pi h'(0) \Omega_3 dx'. \quad (6.14)$$

Similarly to (2.2.27) in [10], we have

$$\partial_{\xi_n} \pi_{\xi_n}^+ \omega^g \sigma_{-1}((D^*)^{-1})(x_0)|_{|\xi'|=1} = -c(\tilde{e}_1) c(\tilde{e}_2) c(\tilde{e}_3) c(dx_n) \left[\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2} \right], \quad (6.15)$$

and

$$\partial_{\xi_n} \partial_{x_n} \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = -\sqrt{-1} h'(0) \left[\frac{c(dx_n)}{|\xi|^4} - 4\xi_n \frac{c(\xi') + \xi_n c(dx_n)}{|\xi|^6} \right] - \frac{2\sqrt{-1} \xi_n \partial_{x_n} c(\xi')(x_0)}{|\xi|^4}. \quad (6.16)$$

Similarly to (6.11), we have

$$\text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \omega^g \sigma_{-1}(D_T^{-1}) \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(D_T^{-1})](x_0) = 0. \quad (6.17)$$

Then

$$\text{Case } \bar{a}) \text{ III}) = \frac{3}{16} \pi h'(0) \Omega_3 dx', \quad (6.18)$$

where Ω_3 is the canonical volume of S^3 .

Thus the sum of Case $\bar{a})$ II) and Case $\bar{a})$ III) is zero.

Case $\bar{b})$: $r = -2$, $\ell = -1$, $k = j = |\alpha| = 0$

By (2.13), we get

$$\begin{aligned} \text{Case } \bar{b}) &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ P^+ \sigma_{-2}((D_T^*)^{-1}) \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-1}) \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad - \frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \omega^g \sigma_{-2}((D_T^*)^{-1}) \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \quad (6.19)$$

Similarly to (3.51), we have

$$-\frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-2}((D_T^*)^{-1}) \partial_{\xi_n} \sigma_{-1}(D_T^{-1})](x_0) d\xi_n \sigma(\xi') dx' = \left[\frac{9}{16} h'(0) - \frac{1}{4} \sum_i A_{iin} \right] \pi \Omega_3 dx'. \quad (6.20)$$

By Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned} \sigma_{-2}((D_T^*)^{-1})(x_0) &= \frac{c(\xi) \sigma_0(D_T^*)(x_0) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[\partial_{x_j}(c(\xi)) |\xi|^2 - c(\xi) \partial_{x_j}(|\xi|^2) \right] (x_0) \\ &= \frac{c(\xi) \sigma_0(D_T^*)(x_0) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[\partial_{x_n}(c(\xi'))(x_0) - c(\xi) h'(0) |\xi'|_{g^{\partial M}}^2 \right]. \end{aligned} \quad (6.21)$$

Hence

$$\pi_{\xi_n}^+ \omega^g \sigma_{-2}((D^*)^{-1})(x_0) := \tilde{A}_1 + \tilde{A}_2, \quad (6.22)$$

where

$$\begin{aligned}\tilde{A}_1 = & c(\tilde{e}_1)c(\tilde{e}_2)c(\tilde{e}_3)c(\mathbf{d}x_n)\left\{-\frac{h'(0)}{2}\left[\frac{c(dx_n)}{4i(\xi_n-i)}+\frac{c(dx_n)-ic(\xi')}{8(\xi_n-i)^2}+\frac{3\xi_n-7i}{8(\xi_n-i)^3}[ic(\xi')-c(dx_n)]\right]\right. \\ & +\frac{-1}{4(\xi_n-i)^2}\left[(2+i\xi_n)c(\xi')\alpha_0c(\xi')+i\xi_nc(dx_n)\alpha_0c(dx_n)+(2+i\xi_n)c(\xi')c(dx_n)\partial_{x_n}c(\xi')\right. \\ & \left.\left.+ic(dx_n)\alpha_0c(\xi')+ic(\xi')\alpha_0c(dx_n)-i\partial_{x_n}c(\xi')\right]\right\};\end{aligned}\quad (6.23)$$

$$\begin{aligned}\tilde{A}_2 = & c(\tilde{e}_1)c(\tilde{e}_2)c(\tilde{e}_3)c(\mathbf{d}x_n)\left\{\frac{-1}{4(\xi_n-i)^2}\left[(2+i\xi_n)c(\xi')\beta_0c(\xi')+i\xi_nc(dx_n)\beta_0c(dx_n)+ic(dx_n)\beta_0c(\xi')\right.\right. \\ & \left.\left.+ic(\xi')\beta_0c(dx_n)\right]\right\}\end{aligned}\quad (6.24)$$

and

$$\alpha_0 = -\frac{3}{4}h'(0)c(dx_n), \quad (6.25)$$

$$\beta_0 = \frac{1}{4}\sum_{i\neq s\neq t}A_{ist}c(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t)-\frac{1}{4}\sum_{i,s,t}\left[-A_{iit}c(\tilde{e}_t)+A_{isi}c(\tilde{e}_s)-A_{iss}c(\tilde{e}_i)+2A_{iii}c(\tilde{e}_i)\right]. \quad (6.26)$$

On the other hand, a simple computation shows

$$\partial_{\xi_n}\sigma_{-1}(D_T^{-1})=\frac{-2i\xi_nc(\xi')}{(1+\xi_n^2)^2}+\frac{i(1-\xi_n^2)c(\mathbf{d}x_n)}{(1+\xi_n^2)^2}. \quad (6.27)$$

From (6.23) and (6.27), we obtain

$$-\frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\text{trace}[\tilde{A}_1\times\partial_{\xi_n}\sigma_{-1}(D_T^{-1})](x_0)\mathbf{d}\xi_n\sigma(\xi')\mathbf{d}x'=0. \quad (6.28)$$

Combining (6.24) and (6.27), we have

$$\text{trace}[\tilde{A}_2\times\partial_{\xi_n}\sigma_{-1}(D_T^{-1})](x_0)=\frac{-i\tilde{c}_0}{2(\xi_n-i)^2(\xi_n+i)^2}, \quad (6.29)$$

where

$$\tilde{c}_0=2(A_{123}-A_{213}+A_{312}). \quad (6.30)$$

From (6.19) and (6.29), we get

$$\begin{aligned}& -\frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\text{trace}[\tilde{A}_2\times\partial_{\xi_n}\sigma_{-1}(D_T^{-1})](x_0)\mathbf{d}\xi_n\sigma(\xi')\mathbf{d}x' \\ & = -\frac{i}{2}\Omega_3\int_{\Gamma^+}\frac{-i\tilde{c}_0}{2(\xi_n-i)^2(\xi_n+i)^2}\mathbf{d}\xi_n\mathbf{d}x' \\ & = -\frac{i}{2}\frac{2\pi i}{1!}\tilde{c}_0\Omega_3\left[\frac{-i}{2(\xi_n+i)^2}\right]^{(1)}|_{\xi_n=i}\mathbf{d}x' \\ & = -\frac{1}{8}\pi\tilde{c}_0\Omega_3\mathbf{d}x'.\end{aligned}\quad (6.31)$$

Combining (6.20) and (6.31), we have

$$\text{Case } \bar{\mathbf{b}}) = \left[\frac{9}{16}h'(0)-\frac{1}{4}(A_{123}-A_{213}+A_{312})-\frac{1}{4}\sum_iA_{iin}\right]\pi\Omega_3\mathbf{d}x'. \quad (6.32)$$

Case $\bar{\mathbf{c}}$: $r=-1, \ell=-2, k=j=|\alpha|=0$

From (2.13), we have

$$\begin{aligned}
\text{Case } \bar{c}) &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ P^+ \sigma_{-1} ((D_T^*)^{-1}) \partial_{\xi_n} \sigma_{-2} (D_T^{-1})] (x_0) d\xi_n \sigma(\xi') dx' \\
&= -\frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1} ((D_T^*)^{-1}) \partial_{\xi_n} \sigma_{-2} (D_T^{-1})] (x_0) d\xi_n \sigma(\xi') dx' \\
&\quad - \frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \omega^g \sigma_{-1} ((D_T^*)^{-1}) \partial_{\xi_n} \sigma_{-2} (D_T^{-1})] (x_0) d\xi_n \sigma(\xi') dx'. \tag{6.33}
\end{aligned}$$

Similarly to (3.63), we have

$$-\frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1} ((D_T^*)^{-1}) \partial_{\xi_n} \sigma_{-2} (D_T^{-1})] (x_0) d\xi_n \sigma(\xi') dx' = \left[-\frac{9}{16} h'(0) - \frac{1}{4} \sum_i A_{iin} \right] \pi \Omega_3 dx'. \tag{6.34}$$

By Lemma 3.5, we have

$$\pi_{\xi_n}^+ \omega^g \sigma_{-1} ((D_T^*)^{-1}) = \left(c(\tilde{e}_1) c(\tilde{e}_2) c(\tilde{e}_3) c(dx_n) \right) \frac{c(\xi') + i c(dx_n)}{2(\xi_n - i)}. \tag{6.35}$$

By Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned}
\sigma_{-2} ((D_T)^{-1}) (x_0) &= \frac{c(\xi) \sigma_0 (D_T) (x_0) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[\partial_{x_j} (c(\xi)) |\xi|^2 - c(\xi) \partial_{x_j} (|\xi|^2) \right] (x_0) \\
&= \frac{c(\xi) \sigma_0 (D_T) (x_0) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[\partial_{x_n} (c(\xi')) (x_0) - c(\xi) h'_{x_n} (0) |\xi'|_{g^{\partial M}}^2 \right]. \tag{6.36}
\end{aligned}$$

Then

$$\partial_{\xi_n} \sigma_{-2} (D_T^{-1}) (x_0) := B_1 + B_2, \tag{6.37}$$

where

$$\begin{aligned}
B_1 &= \frac{1}{(1 + \xi_n^2)^3} \left[(2\xi_n - 2\xi_n^3) c(dx_n) \alpha_0 c(dx_n) + (1 - 3\xi_n^2) c(dx_n) \alpha_0 c(\xi') \right. \\
&\quad + (1 - 3\xi_n^2) c(\xi') \alpha_0 c(dx_n) - 4\xi_n c(\xi') \alpha_0 c(\xi') + (3\xi_n^2 - 1) \partial_{x_n} c(\xi') - 4\xi_n c(\xi') c(dx_n) \partial_{x_n} c(\xi') \\
&\quad \left. + 2h'(0) c(\xi') + 2h'(0) \xi_n c(dx_n) \right] + 6\xi_n h'(0) \frac{c(\xi) c(dx_n) c(\xi)}{(1 + \xi_n^2)^4}, \tag{6.38}
\end{aligned}$$

$$\begin{aligned}
B_2 &= \frac{1}{(1 + \xi_n^2)^3} \left[(2\xi_n - 2\xi_n^3) c(dx_n) \beta_1 c(dx_n) + (1 - 3\xi_n^2) c(dx_n) \beta_1 c(\xi') \right. \\
&\quad \left. + (1 - 3\xi_n^2) c(\xi') \beta_1 c(dx_n) - 4\xi_n c(\xi') \beta_1 c(\xi') \right] \tag{6.39}
\end{aligned}$$

and

$$\beta_1 = \frac{1}{4} \sum_{i \neq s \neq t} A_{ist} c(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t) + \frac{1}{4} \sum_{i,s,t} [-A_{iit} c(\tilde{e}_t) + A_{isi} c(\tilde{e}_s) - A_{iss} c(\tilde{e}_i) + 2A_{iii} c(\tilde{e}_i)]. \tag{6.40}$$

Similar to (6.28), we have

$$-\frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1} \omega^g ((D_T^*)^{-1}) \times B_1] (x_0) d\xi_n \sigma(\xi') dx' = 0. \tag{6.41}$$

From (6.35) and (6.39), we obtain

$$\text{trace}[\pi_{\xi_n}^+ \sigma_{-1} \omega^g ((D_T^*)^{-1}) \times B_2] (x_0) = \frac{i \tilde{c}_0}{(\xi_n + i)^3 (\xi_n - i)}, \tag{6.42}$$

where

$$\tilde{c}_0 = 2(A_{123} - A_{213} + A_{312}). \quad (6.43)$$

By (6.33) and (6.42), we get

$$\begin{aligned} & -\frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1} \omega^g ((D_T^*)^{-1}) \times B_2](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{i}{2} \Omega_3 \int_{\Gamma^+} \frac{i\tilde{c}_0}{(\xi_n + i)^3 (\xi_n - i)} d\xi_n dx' \\ &= -\frac{i}{2} 2\pi i \tilde{c}_0 \Omega_3 \left[\frac{i}{(\xi_n + i)^3} \right]^{(0)}|_{\xi_n=i} dx' \\ &= -\frac{1}{8} \pi \tilde{c}_0 \Omega_3 dx', \end{aligned} \quad (6.44)$$

From (6.34) and (6.44) we obtain

$$\text{Case } \bar{c}) = \left[-\frac{9}{16} h'(0) - \frac{1}{4} (A_{123} - A_{213} + A_{312}) - \frac{1}{4} \sum_i A_{iin} \right] \pi \Omega_3 dx'. \quad (6.45)$$

Now Φ is the sum of the Case \bar{a}) \bar{b}) and \bar{c}), so

$$\Phi = -\frac{1}{2} \left[(A_{123} - A_{213} + A_{312}) + \sum_i A_{iin} \right] \pi \Omega_3 dx'. \quad (6.46)$$

Hence we conclude that

Theorem 6.3. *Let M be a 4-dimensional compact manifold with the boundary ∂M and let \tilde{R} be the scalar curvature of the modified connection $\tilde{\nabla}$. Then we get the volumes associated to $P^+ D^* D$,*

$$\widetilde{Vol}_4^{(1,1)} = -\frac{1}{96\pi^2} \int_M (\tilde{R} \omega^g + \tilde{C}_H) - \frac{1}{2} \int_{\partial M} \left[(A_{123} - A_{213} + A_{312}) + \sum_i A_{iin} \right] \pi \Omega_3 dx'. \quad (6.47)$$

where $\tilde{R} = R + 18 \text{div}(V) - 54|V|^2 - 9 \|T\|^2$ be the scalar curvature of the modified connection $\tilde{\nabla}$ and $\tilde{C}_H = 18(dT - \langle T, *V \rangle \omega^g)$.

7. The Kastler-Kalau-Walze type theorem for 4-dimensional complex manifolds associated with complex nonminimal operators

In this section, we compute the lower dimension volume for lower dimension compact connected manifolds with boundary and get a Kastler-Kalau-Walze type Formula in this case.

Let M be a compact Riemannian manifold of dimension m without boundary. If M is equipped with integrable complex structure, one can split tangential indices into holomorphic and antiholomorphic ones and define space of differential forms $C^\infty(\Lambda^{p,q})$. The exterior differential d can be also split into a sum $d = \partial + \bar{\partial}$ of anticommuting nilpotent operators: $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$. If M is a Kähler manifold, the corresponding ‘‘Laplacians’’ can be reduced to the De-Rham Hodge Laplacian:

$$\partial^* \partial + \partial \partial^* = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial} = \frac{1}{2} \Delta = \frac{1}{2} (d\delta + \delta d). \quad (7.1)$$

Using these first order differential operators one can construct a nonminimal second order differential operator:

$$\mathfrak{D} = g_1 \partial \partial^* + g_2 \partial^* \partial + g_3 \bar{\partial} \bar{\partial}^* + g_4 \bar{\partial}^* \bar{\partial} + g_5 \partial \bar{\partial}^* + g_5^* \bar{\partial} \partial^* \quad (7.2)$$

with real constants g_1, \dots, g_4 and a complex constant g_5 . For some values of the constants this operator reduces to that considered previously in this paper. One can find some motivations for studying nonminimal operators. Such operators appear naturally in quantum gauge theories after imposing gauge conditions.

For complex manifold of dimension 4 with boundary. By Proposition 2.1 in [22], let $D_T = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ we have the identity of three forms $T = \sqrt{-1}(\partial - \bar{\partial})\omega$, where ω be the Kähler forms. In this case, we have $A_{iin} = 0$. Then

Corollary 7.1. *Let M be a 4-dimensional compact complex manifold with the boundary ∂M and $\tilde{\nabla}$ be an orthogonal connection with torsion. Then we get the volumes associated to D_T^2 ,*

$$Vol_4^{(1,1)} = -\frac{1}{48\pi^2} \int_M \tilde{R}(x) dx, \quad (7.3)$$

where $\tilde{R} = R + 18\text{div}(V) - 54|V|^2 - 9 \|T\|^2$ and $\int_M \text{div}(V) dVol_M = -\int_{\partial M} g(n, V) dVol_{\partial M}$.

Next for the operator $(\bar{\partial} + \bar{\partial}^*)^2$, we consider the heat kernel for nonminimal operators acting on the space $C^\infty(\Lambda^k)$ of k forms. Let us discuss the first order operators $D_1 = \partial$ and $D_2 = \bar{\partial}$ which satisfy the properties of Lemma 1. And these operators will be used to build up the following general nonminimal second order operator

$$\mathfrak{D} = a^2 \bar{\partial} \bar{\partial}^* + b^2 \bar{\partial}^* \bar{\partial}. \quad (7.4)$$

The nonminimal operator with real constants a^2, b^2 is the most general hermitian operator on $C^\infty(\Lambda^k)$ which can be constructed using $\partial, \bar{\partial}, \partial^*$ and $\bar{\partial}^*$. This operator has the form (7.2). Denote by $\sigma_l(\mathfrak{D})$ the l -order symbol of an operator \mathfrak{D} . We compute the symbol expansion of $\mathfrak{D} = a^2 \bar{\partial} \bar{\partial}^* + b^2 \bar{\partial}^* \bar{\partial}$. Recall [20], we have

Lemma 7.2. *The following equalities hold*

$$\sigma_L(\bar{\partial})(x, \xi) = \frac{\sqrt{-1}}{2} \sum_{1 \leq j \leq n} (\xi_j + \sqrt{-1} \xi_{j+n}) \varepsilon(e^j - \sqrt{-1} e^{j+n}); \quad (7.5)$$

$$\sigma_L(\bar{\partial}^*)(x, \xi) = -\frac{\sqrt{-1}}{2} \sum_{1 \leq j \leq n} (\xi_j - \sqrt{-1} \xi_{j+n}) \iota(e^j - \sqrt{-1} e^{j+n}); \quad (7.6)$$

$$\sigma_L(\tilde{\Delta}) = \frac{1}{2} |\xi|^2 \text{Id}, \quad (7.7)$$

where $\tilde{\Delta} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$.

For $\xi = \sum_{1 \leq i \leq 2n} \xi_i e^i$, let $\hat{\xi} = \sum_{1 \leq j \leq n} (\xi_j + \sqrt{-1} \xi_{j+n})(e^j - \sqrt{-1} e^{j+n})$. For a $(0, 1)$ -form ω_1 , we have

$$\begin{aligned} \iota^{g_i}((\xi_j + \sqrt{-1} \xi_{j+n})(e^j - \sqrt{-1} e^{j+n})) \omega_1 &= \langle \omega_1, (\xi_j + \sqrt{-1} \xi_{j+n})(e^j - \sqrt{-1} e^{j+n}) \rangle \\ &= (\xi_j - \sqrt{-1} \xi_{j+n}) \iota^{g_i}(e^j - \sqrt{-1} e^{j+n}) \omega_1. \end{aligned} \quad (7.8)$$

By Lemma 7.2 and (7.8), we have

$$\sigma_L(\bar{\partial})(\xi) = \frac{\sqrt{-1}}{2} \varepsilon(\hat{\xi}); \quad \sigma_L(\bar{\partial}^*)(\xi) = -\frac{\sqrt{-1}}{2} \iota(\hat{\xi}). \quad (7.9)$$

Then we obtain the following Lemma.

Lemma 7.3. *Let $\mathfrak{D} = a^2 \bar{\partial} \bar{\partial}^* + b^2 \bar{\partial}^* \bar{\partial}$ on $C^\infty(\Lambda^k)$, then*

$$\sigma_{-2}(\mathfrak{D}^{-1}) = \frac{2b^2 |\xi|^2 + (a^2 - b^2) \iota(\hat{\xi}) \varepsilon(\hat{\xi})}{a^2 b^2 |\xi|^4}. \quad (7.10)$$

Proof. From Lemma 7.2, we have $\sigma_2(\overline{\partial\partial^*}) = \frac{1}{4}\varepsilon(\hat{\xi})\iota(\hat{\xi})$, $\sigma_2(\overline{\partial\partial^*} + \overline{\partial^*\partial}) = \frac{1}{2}|\xi|^2$. Combining these results, we obtain

$$\sigma_2(a^2\overline{\partial\partial^*} + b^2\overline{\partial^*\partial}) = (a^2 - b^2)\sigma_2(\overline{\partial\partial^*}) + b^2\sigma_2(\overline{\partial\partial^*} + \overline{\partial^*\partial}) = \frac{(a^2 - b^2)}{4}\varepsilon(\hat{\xi})\iota(\hat{\xi}) + \frac{b^2}{2}|\xi|^2. \quad (7.11)$$

An application of $2|\xi|^2 = \varepsilon(\hat{\xi})\iota(\hat{\xi}) + \iota(\hat{\xi})\varepsilon(\hat{\xi})$ shows

$$\left[\frac{(a^2 - b^2)}{4}\varepsilon(\hat{\xi})\iota(\hat{\xi}) + \frac{b^2}{2}|\xi|^2\right]\left[\frac{(a^2 - b^2)}{4}\iota(\hat{\xi})\varepsilon(\hat{\xi}) + \frac{b^2}{2}|\xi|^2\right] = \frac{a^2b^2}{4}|\xi|^4. \quad (7.12)$$

Then

$$\sigma_{-2}(\mathfrak{D}^{-1}) = \frac{2b^2|\xi|^2 + (a^2 - b^2)\iota(\hat{\xi})\varepsilon(\hat{\xi})}{a^2b^2|\xi|^4}. \quad (7.13)$$

□

Without loss of generality, we may assume the differential operator $\mathfrak{D} = a^2\overline{\partial\partial^*} + b^2\overline{\partial^*\partial}$ acting on $C^\infty(\Lambda^k)$. An application of Theorem 1 in [23] yields the following:

Theorem 7.4. [23] *For m -dimensional ($m > 2$) compact Kähler manifolds without boundary and the associated nonminimal operator $\mathfrak{D} = a^2\overline{\partial\partial^*} + b^2\overline{\partial^*\partial}$ on $C^\infty(\Lambda^k)$, then*

$$\begin{aligned} a_2[\mathfrak{D}|_{C^\infty(\Lambda^k)}] &= \left(\frac{a^2}{2}\right)^{2-\frac{m}{2}} \sum_{p=0}^{k-2} (-1)^{k-p}(k-p-1)a_2(\Delta_p) \\ &\quad - \left[\left(\frac{a^2}{2}\right)^{2-\frac{m}{2}} + \left(\frac{b^2}{2}\right)^{2-\frac{m}{2}}\right] \sum_{p=0}^{k-1} (-1)^{k-p}(k-p)a_2(\Delta_p) \\ &\quad + \left(\frac{b^2}{2}\right)^{2-\frac{m}{2}} \sum_{p=0}^k (-1)^{k-p}(k-p+1)a_2(\Delta_p). \end{aligned} \quad (7.14)$$

From the Theorem 4.8.18 in [24] we obtain

Theorem 7.5. [24] *Let the $\Delta_p^m = d\delta + \delta d$ denote the Laplacian acting on the space of smooth p -forms on an m -dimensional manifold. We let R_{ijkl} denote the curvature tensor with the sign convention that $R_{1212} = -1$ on the sphere of radius 1 in R^3 . Then:*

$$a_0(\Delta_p^m) = (4\pi)^{-\frac{m}{2}} C_m^p; \quad (7.15)$$

$$a_2(\Delta_p^m) = \frac{(4\pi)^{-\frac{m}{2}}}{6} \left(C_{m-2}^{p-2} + C_{m-2}^p - 4C_{m-2}^{p-1} \right) (-R_{ijij}). \quad (7.16)$$

Then for $m \geq 4$ the coefficients are

$$a_2(\Delta_0^m) = \frac{(4\pi)^{-\frac{m}{2}}}{6} (-R_{ijij}); \quad (7.17)$$

$$a_2(\Delta_1^m) = \frac{(4\pi)^{-\frac{m}{2}}}{6} (6-m)(-R_{ijij}); \quad (7.18)$$

$$a_2(\Delta_2^m) = \frac{(4\pi)^{-\frac{m}{2}}}{12} (-m^2 + 13m - 24)R_{ijij}; \quad (7.19)$$

$$a_2(\Delta_3^m) = \frac{(4\pi)^{-\frac{m}{2}}}{36} (108 - 92m + 21m^2 - m^3)R_{ijij}; \quad (7.20)$$

$$a_2(\Delta_4^m) = \frac{(4\pi)^{-\frac{m}{2}}}{144} (-576 + 630m - 227m^2 + 30m^3 - m^4)R_{ijij}. \quad (7.21)$$

Let $k = 4$ (the calculation of other case similar), combining these results, we obtain the main theorem of this section.

Theorem 7.6. *For 4-dimension compact Kähler manifolds without boundary and the associated nonminimal operator $\mathfrak{D} = a^2 \bar{\partial} \partial^* + b^2 \partial^* \bar{\partial}$ on $C^\infty(\Lambda^4)$, then*

$$\begin{aligned} Wres(\mathfrak{D}^{-1}) &= \frac{(m-2)(2\pi)^4}{\Gamma(\frac{m-k}{2})} \left\{ \left[\left(\frac{a^2}{2} \right)^{2-\frac{m}{2}} - \left(\frac{b^2}{2} \right)^{2-\frac{m}{2}} \right] \frac{(4\pi)^{-\frac{m}{2}}}{36} (222 - 167m + 24m^2 - m^3) R_{ijij} \right. \\ &\quad \left. + \left(\frac{b^2}{2} \right)^{2-\frac{m}{2}} \frac{(4\pi)^{-\frac{m}{2}}}{144} (-576 + 630m - 227m^2 + 30m^3 - m^4) R_{ijij} \right\} \Big|_{m=4} \\ &= -\frac{\pi^2}{3} R_{ijij}, \end{aligned} \tag{7.22}$$

where R_{ijkl} denote the curvature tensor with the sign convention that $R_{1212} = -1$ on the sphere of radius 1 in R^3 .

Acknowledgements

This work was supported by Fok Ying Tong Education Foundation under Grant No. 121003 and NSFC. 11271062. And Jian Wang's Email address: wangj068@gmail.com. The author also thank the referee for his (or her) careful reading and helpful comments.

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